12 The Wave Equation

The wave equation in one space and one time variable is

$$\frac{d^2u}{dt^2} - \frac{d^2u}{dx^2} = 0, \qquad u : (\mathbb{R}_x, \mathbb{R}_t) \to \mathbb{R}$$

Theorem. *If $f_0, f_1 \in C^\infty(\mathbb{R}_x)$, $\exists! u \in C^\infty(\mathbb{R}^2_{x,t})$ satisfying the wave equation with the initial conditions $u(0,x) = f_0(x)$ and $\partial_x u(0,x) = f_1(x)$, $\forall x \in \mathbb{R}$.*

Proof. We can write all solutions of the wave equation as $u(t,x) = \varphi(t+x) + \psi(t-x)$ where $u \in C^\infty(\mathbb{R}^2)$ and $\varphi, \psi \in C^\infty(\mathbb{R})$.

If $u(t,x) = \varphi(t+x) + \psi(t-x)$ obviously

$$\frac{\partial^2}{\partial t^2}\varphi(t+x) = \frac{\partial^2}{\partial x^2}\varphi(t-x)$$

Conversely if we have $\frac{d^2u}{dt^2} - \frac{d^2u}{dx^2} = 0, \qquad u : (\mathbb{R}_x, \mathbb{R}_t) \to \mathbb{R}$ then consider

$$v(t,x) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}$$

then

$$\frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial t \partial x}, \qquad \frac{\partial v}{\partial x} = \frac{\partial^2 u}{\partial t \partial x} + \frac{\partial^2 u}{\partial x^2}$$

then by the equality of mixed derivatives, $\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x}$. If we define $v(t,x) = \varphi(s,r)$ where $s = t+x$ and $r = t-x$, then

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial \varphi}{\partial s}\frac{\partial s}{\partial t} + \frac{\partial \varphi}{\partial r}\frac{\partial r}{\partial t} = \frac{\partial \varphi}{\partial s} + \frac{\partial \varphi}{\partial r} \\ \frac{\partial v}{\partial x} &= \frac{\partial \varphi}{\partial s}\frac{\partial s}{\partial x} - \frac{\partial \varphi}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial \varphi}{\partial s} - \frac{\partial \varphi}{\partial r} \end{aligned}$$

if $\partial v/\partial t = \partial v/\partial x$ then $\partial\varphi/\partial r = 0$. So φ is only a function of s, so it is a function of $t+x$. So $v = \partial u/\partial t + \partial u/\partial x = \varphi(t+x)$. If $\Phi' = \varphi$ then $u = \frac{1}{2}\Phi(t+x)$ satisfies that equation. So if we let $u = \varphi_1(t+x) + u'$ then $\partial u'/\partial t + \partial u'/\partial x = 0$. If we apply the same argument with the signs switched then $u' = \psi(t-x)$, and so $u = \varphi_1(t+x) - \psi(t-x)$.

Now the initial conditions:

$$\begin{aligned} u(0) &= \varphi(x) + \psi(-x) = f_0 \\ \frac{\partial u}{\partial t}(0,x) &= \varphi'(x) + \psi'(-x) = f_1 \end{aligned}$$

If we differentiate the first equation we get $\varphi'(x) - \psi'(-x) = f_0'$ then

$$\varphi'(x) = \frac{1}{2}(f_1 + f_0') \qquad \psi'(x) = \frac{1}{2}(f_1(-x) - f_0'(-x))$$

If we take the following it solves the equation

$$\varphi(x) = \int_0^x \frac{1}{2}(f_1 + f_0')dx$$

$\square$

Exercise $\varphi + \psi$ is not unique, but u is unique.

12.1 Fourier Series solutions

From the uniqueness of u if f_0 and f_1 are both 2π-periodic then

$$f_i(x+2\pi) = f_i(x),\ \forall x \in \mathbb{R} \Longrightarrow u(t, x+2\pi) = u(t,x),\ \forall t, x \in \mathbb{R}^2$$

Because the equation is translation invariant, $\tilde{u}(t,x) = u(t, x+2\pi)$ satisfies

$$\frac{\partial^2 \tilde{u}}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial t^2}(t, x+2\pi), \qquad \frac{\partial^2 \tilde{u}}{\partial x^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t, x+2\pi)$$

$\tilde{u}(0,x) = f_0(x+2\pi)$.

Now, we can expand the solution in the fourier series

$$\begin{cases} u(t,x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} c_k(t)e^{ixk} \\ c_k(t) = \int_{-\pi}^{\pi} u(t,x)e^{-ixk}dx \end{cases}$$

when we combine our wave equation condition with $\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2$ with the below, we get

$$\frac{d^2 c_k(t)}{dt^2} = \int_{-\pi}^{\pi} \frac{\partial^2 u}{\partial t^2}(t,x)e^{-ixk}dx = \int_{-\pi}^{\pi} \frac{\partial^2 u}{\partial x^2}(t,x)e^{-ixk}dx$$

Integrate by parts (there are no boundary terms by periodicity) and this is

$$\int_{-\pi}^{\pi} u(t,x)\frac{d^2}{dx^2}e^{-ixk}dx = -k^2\int_{-\pi}^{\pi} u(t,x)e^{-ixk}dx = -k^2 c_k(t)$$

So the Fourier coefficients of $u(t,x)$ with f_0, f_1 are 2π-periodic (initial conditions/driving system 2π periodic) satisfy

$$\left(\frac{d^2}{dt^2} + k^2\right)c_k(t) = 0$$

So the general solution for the Fourier coefficients are of the form $c_k(t) = a_k e^{itk} + b_k e^{-itk}$. So if the initial data is 2π-periodic and smooth, then the solutions are 2π periodic in t (as well as space).

Now to solve for a_k, b_k:

$$c_k(0) = \int_{-\pi}^{\pi} u(0,x)e^{-ixk}dx = \int_{-\pi}^{\pi} f_0(x)e^{-ixk}dx = c_k(f_0)$$
$$c_k'(0) = \frac{d}{dt}c_k(0) = \int_{-\pi}^{\pi} \frac{du}{dt}(0,x)e^{-ixk}dx = \int_{-\pi}^{\pi} f_1 e^{-ixk}dx = c_k(f_1)$$

but $c_k(0) = a_k + b_k$ and $c_k'(0) = ia_k - ib_k$, and so the coefficients are

$$a_k = \frac{1}{2}(c_k(0) + \frac{1}{ik}c_k'(0)), \qquad b_k = \frac{1}{2}(c_k(0) - \frac{1}{ik}c_k'(0))$$

and for $k = 0$, $c_k(0) = a_0$ and $c_k'(0) = b_0$. And the solution is 2π-periodic in t if and only if $b_0 = \int_{-\pi}^{\pi} f_1(x)dx = 0$.

So now we have 2 ways of solving the equation for 2π-period input data

1. By using $u(t,x) = \varphi(t+x) + \psi(t-x)$.
2. If we set
$$\alpha_k = \int_{-\pi}^{\pi} f_0 e^{-ixk}dx, \qquad \beta_k = \int_{-\pi}^{\pi} f_1 e^{-ixk}dx$$
then the Fourier series solution is

$$u(t,x) = \frac{1}{2\pi}\sum_{k\neq 0}\left[\underbrace{\frac{1}{2}(\alpha_k + \frac{1}{ik}\beta_k)}_{a_k} e^{i(t+x)k} + \underbrace{\frac{1}{2}(\alpha_k - \frac{1}{ik}\beta_k)}_{b_k} e^{i(x-t)k}\right] + \underbrace{\frac{1}{2\pi}(\alpha_0 + \beta_0 t)}_{k=0 \text{ term}}$$

Note: k^2 are the eigenvalues of $\frac{d^2}{dx^2}$ on $[-\pi, \pi]$.